PHYS 211: Physics for Science & Engineering I (Summer 2023)
(Lecture Notes)
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ACKNOWLEDGMENTS

Contents

Chapter 1

Vector mathematics

1.1 SI units

In this course, we will mainly follow SI units. We will concern ourselves with three fundamental quantities, namely, mass, length and time, although there are seven. A combination of fundamental quantities gives us derived quantities: speed = length/meters. The fundamental SI units of mass, length, and time are kilograms (kg), meters (m), and second (s). A combination of fundamental units gives us derived units.

$$N = \frac{kg\ m}{s^2} \qquad \text{Newton: unit of force,} \qquad (1.1)$$

$$J = \frac{kg\ m^2}{s^2} \qquad \text{Joule: unit of energy} \qquad (1.2)$$

When fundamental quantities combine to give a derived quantity, their units combine the same way to give the unit of the derived quantity. As an example, unit of speed is m/s.

1.2 Scalars and vectors

Hand-wavy definition: A vector has a magnitude (with a unit) and a direction. A scalars has no direction. For our purposes, a scalar is a real number accompanied by a unit. Vectors are represented by an alphabet with an arrow above it (such as $\vec{V}$). Scalars are represented by simply an alphabet without the arrow.

Magnitude of a scalar: Magnitude of a scalar c is its absolute value (with the units). It is denoted by $|c|$. For example $|-4N| = 4N$.

1.2.1 More definitions in the context of vectors

A vector is composed on its magnitude (including the unit) and direction. The magnitude of a vector $\vec{V}$ (denoted by $|\vec{V}|$ or V) is a non-negative real number (with units), and serves to represent how "big/strong/long" the vector is. Note that V (magnitude of $\vec{V}$) is a scalar since it has no direction information. This justifies denoting it with an alphabet without an arrow.

Therefore two vectors are defined to be equal if and only if their magnitudes and directions are the same. A zero vector ($\vec{0}$ m/s or $\vec{0}$ N) is defined to have zero magnitude and an undefined direction. Negative of a vector $\vec{V}$ is defined to be another vector (denoted by $-\vec{V}$), which has the same magnitude as the $\vec{V}$, but opposite direction. Thus, $|\vec{V}| = |-\vec{V}| = V$.

1.2.2 Graphical representation of vectors

A vector $\vec{V}$ in a certain direction is represented by an arrow in the same direction. The length of the arrow is in proportion to V, decided by a scaling. For example, if $\vec{V}$ is a force vector towards

the south direction such that $V = 30$ Newtons (N), and if for drawing purposes, we decide a scaling 1 cm $\sim$ 10 N, then $\vec{V}$ is represented by an arrow, which is 3 cm long, and points in the south direction.

Caution: This drawing is just a representation of the vector; it is not the vector itself. It is nonetheless very useful because it helps us visualize the vector, and perform algebraic operation with it (addition, subtraction, taking components, as we will soon see).

1.2.3 Vector addition and subtraction: graphical approach

The graphical procedure to add a number of vectors (*with the same units*) goes as follows

- Draw any vector.

- Draw the second (*order does not matter*) vector such that its tail is at the head of the first vector.

- Draw the next vector such that its tail is at the head of the previous vector. Repeat the procedure until all vectors have been exhausted.

- Join the tail of the first vector with the head of the last vector with a straight line. This represents the sum of all the vectors, and it points from the tail of the first vector to the head of the second vector.

Vector subtraction: $\vec{A} - \vec{B} \equiv \vec{A} + -\vec{B}$.[1]

1.2.4 Multiplication and division of vectors with scalars

Product of a vector $\vec{V}$ and a scalar c is another vector $c\vec{V}$, such that $|c\vec{V}| = |c||\vec{V}| = |c|V$. The direction of $c\vec{V}$ is same as $\vec{V}$ if $c > 0$ (units) , opposite to $\vec{V}$ if $c < 0$ (units), and undefined when $c = 0$ (units), for $c = 0$ would render $c\vec{V}$ a zero vector (which has no defined direction).

Division: $\vec{V}/c \equiv (1/c)\vec{V}$.

1.2.5 Caution about notation

- $|c|$ denotes the magnitude (absolute value of a scalar), whereas $|\vec{V}|$ denotes the operation of extracting the magnitude of a vector. The result in both the cases is a non-negative real number (with units).

- A scalar is generally denoted by a alphabet without an arrow; e.g. *c*. *c* could be a negative number (with units). But $V \equiv |\vec{V}|$ is a special scalar (hence denoted by an alphabet without an arrow) which is by definition a non-negative number (with units). So, an alphabet without an arrow may represent a scalar with either sign.

1.2.6 Laws of vector addition and multiplication

- Commutativity (addition): $\vec{A} + \vec{B} = \vec{B} + \vec{A}$.

- Associativity (addition): $\vec{A} + \left(\vec{B} + \vec{C}\right) = \left(\vec{A} + \vec{B}\right) + \vec{C}$.

- Distributivity (multiplication with a scalar): $c(\vec{A} + \vec{B}) = c\vec{A} + c\vec{B}$.

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[1]The symbol "$\equiv$" stands for "is defined as".

1.3 Describing vector using their components

1.3.1 Unit vectors

For a certain vector $\vec{V}$, we can associate a vector $\hat{V} \equiv \vec{V}/V$. It's easy to check that its magnitude is 1 (no units), and it points in the same direction as $\vec{V}$. Such a vector is called a unit vector, and is denoted by a hat, rather than an arrow. One can define a unit vector $\hat{U}$ directly, without invoking an associated non-unit vector $\vec{U}$.

1.3.2 Components of a vector

A set of coordinate axes is formed by attaching two real number axes (x and y axes) perpendicularly. Here is how we we operationally define the coordinates of the head of a vector (with the vector placed so that its tail is at the origin of the axes)

- From the head of a vector $\vec{V}$ draw two new lines, which are perpendicular to the two axes.

- The real numbers associated with the point on the two axes where these new perpendiculars meet the axes are the corresponding coordinates of the head of the vector.

- The x and y components (denoted by V_x and V_y) of $\vec{V}$ are defined to be the x and y coordinates of the head of $\vec{V}^2$.

1.3.3 The component method of specifying a vector

It's easy to convince oneself that a vector $\vec{V}$ is equally well be defined by stating its components (V_x, V_y), rather than the magnitude-direction pair, as we have been doing till now. More concretely, $\vec{V} = V_x\hat{i} + V_y\hat{j}$, where the two unit vectors $\hat{i}$ and $\hat{j}$ are the unit vectors in the direction of x- and y-axes. Verify the above two statements by means of drawing pictures. Some remarks are in order.

- (V_x, V_y) are scalars, and hence they are denoted without arrows. However they not magnitudes of any vector, and hence can be negative.

- Changing the coordinate axes alone changes the components of a given vector.

1.3.4 Quadrants, angles and trigonometry

Quadrants

The point (x, y) is said to lie in

- Quadrant 1: $x > 0$; $y > 0$.

- Quadrant 2: $x < 0$; $y > 0$.

- Quadrant 3: $x < 0$; $y < 0$.

- Quadrant 4: $x > 0$; $y < 0$.

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^{2}Coordinates are associated with a point, whereas components are associated with a vector

Angle conventions

- When we talk about the angle θ between two lines, where the none of the lines is a coordinate axis, then $0° \leq \theta \leq 180°$. We can refer to this as unsigned angle for this course.[3]

- When one of the lines is the $+x$-axis, and to get to the other line, one sweeps towards the $+y$-axis, then $\theta > 0°$. If one sweeps towards the $-y$-axis, then $\theta < 0°$. We can call this the signed angle for our course.

We are expected to tell whether we are talking about signed or unsigned angle from the context.

Trigonometry

What's the definition of trigonometric functions of an angle θ? Draw a line (vector) with its tail at the origin of a coordinate axis, such that one has to sweep an angle θ from the $+$ve x-axis to reach this vector. The sweep is counter-clockwise if $\theta > 0$, and vice versa. Denote the coordinates of the head (a point) of the vector by (x, y). Also define $r \equiv \sqrt{x^2 + y^2}$ (the length of the arrow). Then

$$\sin \theta = \frac{y}{r}, \tag{1.3}$$

$$\cos \theta = \frac{x}{r}, \tag{1.4}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{y}{x}. \tag{1.5}$$

$$\tag{1.6}$$

Signed angle of a line or a vector

Now it is easy to see from the above definition that $\tan \theta = \tan \theta + n\ 180°$, where n can be any integer. And because of this, extracting the angle from its tan is dubious (a very important point). This is reflected in the fact that $\arctan(\tan 30°) = 30°$, whereas $\arctan(\tan 120°) = -60°$. This can be explained by noting that arctan gives values in the range $[-90°, 90°]$, by definition.

But there is an easy way to determine the (signed) angle made by a line (or rather a vector) and the $+$ve x-axis from its components

- From its components, determine the quadrant it is in. It is easy.

- Evaluate $\arctan y/x$. Either this is the correct angle, or it is off by $\pm 180°$ (because of the above "dubiousness").

- If the angle $\arctan y/x$ corresponds to the correct quadrant (remember we know the correct quadrant from the first step), then we have the correct signed angle. Otherwise, just add or subtract $180°$, which gives us $\arctan y/x \pm 180°$.

1.3.5 Connection between the magnitude-direction and the component description of vectors

One can specify the magnitude V of a vector $\vec{V}$, and its direction via the signed angle θ. How do we get its components? To switch from the magnitude-direction description of vectors to the component one, we use

$$V_x = V \cos \theta; \qquad V_y = V \sin \theta. \tag{1.7}$$

This easily follows from the definition of trigonometric functions given above.

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[3]Beware, it is not a standard terminology.

Going from the component description to the magnitude-direction description is a little tricky for the reasons discussed above. It is encapsulated in (with $V \equiv |\vec{V}|$ and θ being the signed angle of $\vec{V}$)

$$V = \sqrt{V_x^2 + V_y^2}; \qquad \theta = \arctan \frac{V_y}{V_x} + \alpha, \tag{1.8}$$

where $\alpha = 0$, if $\vec{V}$ lies in the first or the fourth quadrant; otherwise $\alpha = \pm 180°$.

1.4 Vector math in component language

- Two vectors $\vec{A}$, and $\vec{B}$ are equal if and only if all their respective components are equal: $A_x = B_x$; $A_y = B_y$.

- The zero vector has both components equal to 0.

- If $\vec{A} = -\vec{B}$, then $A_x = -B_x$; $A_y = -B_y$.

- If $\vec{C} = \vec{A} + \vec{B}$, then $C_x = A_x + B_x$; $C_y = A_y + B_y$. Similarly, if $\vec{C} = \vec{A} - \vec{B}$, then $C_x = A_x - B_x$; $C_y = A_y - B_y$.

- If $\vec{B} = c\vec{A}$, (c is a scalar) then $B_x = cA_x$; $B_y = cA_y$. Similarly, if $\vec{B} = \vec{A}/c$, then $B_x = A_x/c$; $B_y = A_y/c$.

Remembering the concepts of equality, zero vector, negatives, addition/subtraction and multiplication/division in the context of vectors (presented in Sec. 1.2 in the language of magnitude and direction), one can easily verify these rules pictorially.

1.5 Scalar or dot product of two vectors

We will confine ourselves to two dimensions. Dot product of two vectors is defined as

$$\vec{A} \cdot \vec{B} = AB \cos \phi, \tag{1.9}$$

where ϕ is the (unsigned) angle between the two vectors so that $0° \le \phi \le 180°$. Angle conventions were laid out in Sec. 1.3.4. It follows that the dot product between perpendicular vectors is zero.

It is easy to see that $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = 1$, whereas $\hat{i} \cdot \hat{j} = 0$. Assuming that the law of distributivity holds for dot products, it is also easy to show (using the above results) that

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y. \tag{1.10}$$

1.6 Exercises

1. Consider the vectors in Fig. 1.1. (i) Via graphical method (using scale drawings) find $\vec{A} + \vec{B}, \vec{A} - \vec{B}, -\vec{A} + \vec{B}, -\vec{A} - \vec{B}$. (ii) Verify the consistency of the first result with the last, and the second with the third.

2. Find the magnitude and direction (via signed angle) of the following vectors (i) $(V_x, V_y) = (-3, -2)$ (ii) $(V_x, V_y) = (3, -2)$ (ii) $(V_x, V_y) = (-3, 2)$.

3. Redo Question 1 using component method and verify the results with the ones obtained using graphical method.
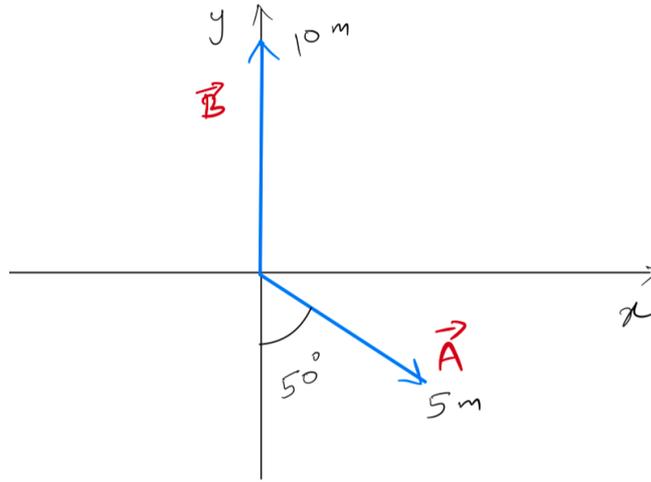
Figure 1.1:

4. A postal employee charts out a trajectory as shown in Fig. 1.2 (i) Find the magnitude and direction (using signed angle) of the final displacement using component method.[4] (ii) The employee then retraces their steps back to return to the starting point. Write out the vectors associated with this return journey in component form. (iii) Add all the vector (initial and return journey). Is this resultant consistent with the fact that after the return journey, the employee is right where they started from?

5. In the context of Fig. 1.1, find the magnitude and direction of $\vec{C} = 4\vec{A} - 2\vec{B}$.

6. (i) Is $\vec{W} \equiv (\hat{i} + \hat{j})$ a unit vector?[5] (ii) We want to find a scalar c such that $c\vec{W}$ is a unit vector. Find c.

7. Two forces act on the red object as shown in Fig. 1.3[6]. (i) What is the resultant force on the object? (ii) Find the third force, which when applied to this object (in addition to the two forces in the figure) would lead to no acceleration.[7]

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[4]Displacement of a certain object obeys vector mathematics.
[5]A vector is a unit vector if it has magnitude equal to 1 (and no units).
[6]Forces obey vector mathematics.
[7]We will learn in a later chapter on Newton's laws that zero net force leads to zero acceleration.
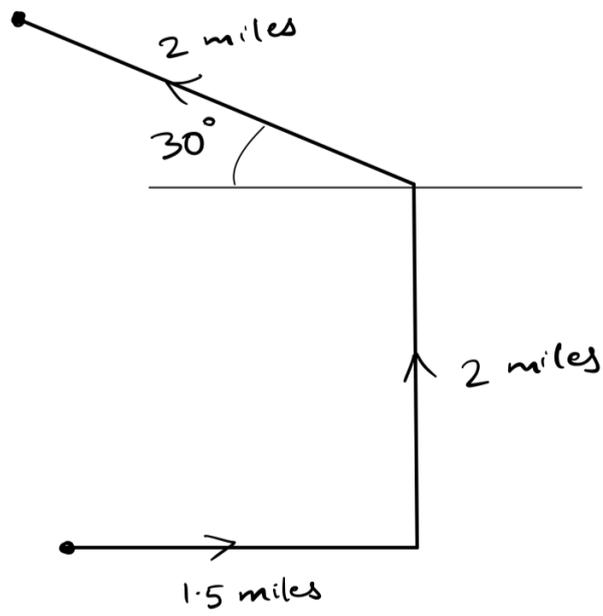
Figure 1.2:
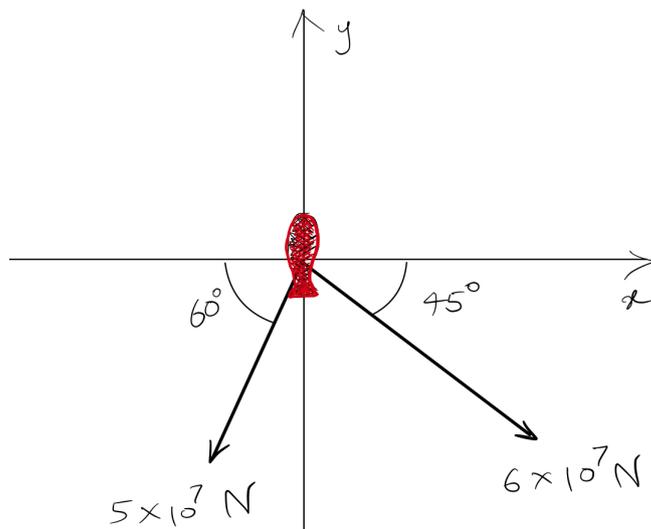
Figure 1.3:

Chapter 2

Kinematics in one dimension

2.1 Some basics

2.1.1 Using components to represent vectors

When all the vectors under consideration point along or opposite to a certain direction, then we can specify the vectors using only their components along an axis that point along this direction. This is why we will work only with components of the vectors in this chapter.

2.1.2 Change

Change in a certain quantity f is usually denoted by Δf, and is the difference between the final and the initial values of f, i.e. $\Delta f = f_\mathrm{f} - f_\mathrm{i}$. Note that infinitesimal changes will be denoted by df.

2.2 Introducing kinematical quantities

Position vector: The position vector of a point (with coordinates x and y) wrt. a coordinate system[1] is

$$\vec{r} = x\hat{i} + y\hat{j}. \tag{2.1}$$

Note that $\vec{r}$ depends on the coordinate system.

Displacement vector: Displacement vector is the change in the position vector, i.e.

$$\Delta\vec{r} = \vec{r}_\mathrm{f} - \vec{r}_\mathrm{i}. \tag{2.2}$$

Interestingly, $\Delta\vec{r}$ is the same in two frames, which are stationary (neither moving not rotating) wrt. each other, although $\vec{r}$ depends on the frame.

Distance traveled: It is a very basic quantity, which even mathematically unsophisticated minds have an idea about. However it does warrant a technical-looking definition involving calculus. Break up the total journey of the point in consideration into several small segments, each of which corresponds to a certain tiny displacement vector $\Delta\vec{r}$ or $d\vec{r}$. The distance traveled is defined as

$$\Delta s = \sum |\Delta\vec{r}| = \int |d\vec{r}|. \tag{2.3}$$

Average velocity vector:

$$\vec{v}_\mathrm{avg} = \frac{\Delta\vec{r}}{\Delta t}. \tag{2.4}$$

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[1]We may also use the word "frame of reference" or just "frame" for coordinate system.

Speed: (Instantaneous) speed is $|\vec{v}|$. Average speed is distance traveled divided by the time taken, or $\Delta s/\Delta t$. Exercise 1 of this chapter will help you see that average speed is not the magnitude of the average velocity, although speed is defined to be the magnitude of the velocity.

The table below helps us see why describing a vector in terms of its components is fine, if this vector is restricted along a certain fixed direction (say x-axis)

Position	$x\hat{i}$	x
Displacement	$(x_f - x_i)\hat{i}$	$x_f - x_i$
Average velocity	$\dfrac{x_f - x_i}{t_f - t_i}\hat{i}$	$\dfrac{x_f - x_i}{t_f - t_i}$

2.2.1 One dimensional kinematical quantities in component form

Position is denoted by the coordinate of the point x. Displacement is $\Delta x = x_f - x_i$. Average velocity is $v_{\mathrm{av}_x} = \Delta x/\Delta t$. (Instantaneous) velocity is $v_x = dx/dt$. Average acceleration is $a_{\mathrm{av}_x} = \Delta v_x/\Delta t$. (Instantaneous) acceleration is $a_x = dv_x/dt$. It is understood here that we are referring to vectors via their components, for in true sense $\Delta x = x_f - x_i$, is not the displacement vector but rather a component of it.

2.3 Digression into mathematics

First of all, the roots of a quadratic equation $ax^2 + bx + c = 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2.5}$$

On a y vs x graph, $\Delta y/\Delta x = (y_f - y_i)/(x_f - x_i)$ is found computing rise/run with a triangle, two of whose vertices correspond to the initial and final points. Usually we make our run > 0, by having $x_f > x_i$. But rise will be positive or negative depending on whether $y_f - y_i > 0$, or vice-versa.

Now dy/dx at a certain point in the graph is found by again applying rise/run on the tangent to that point; the associated triangle can be of arbitrary size. dy/dx is also equal to the tan of the signed angle that the above tangent forms with the +ve x-axis. dy/dx is also called the slope of the graph.

We mention two important calculus facts. First, recall that $dy/dx = nx^{n-1}$ if $y = x^n$. And finally at a maximum of a function $y(x)$, $dy/dx = 0$; $d^2y/dx^2 < 0$, and at a mininum $dy/dx = 0$; $d^2y/dx^2 < 0$.

2.4 Explanation with figures

Discuss using Figs. 2.8 and 2.14 of Ref. [1], and the Mathematica notebook. How to physically replicate the graphs?

2.4.1 Speeding up or slowing down

A body is said to speeds up or slow down when the associated $v = |\vec{v}|$ increases or decreases. Convince yourself by drawing figures that the former is true when $v_x a_x > 0$ (v_x and a_x have the same sign), and the latter is true when $v_x a_x < 0$ (opposite signs). The sign of a_x alone does not tell whether the object is speeding up or slowing down.

2.5 Kinematical equations (constant acceleration)

Notation: Up until now, we used the subscript i and f to denote the initial and final versions of quantities. However we will also use 0 to denote the initial version of a quantity, whereas no subscript will imply the quantity's final value.

We usually set our clocks so that $t_i = 0 \implies \Delta t = t - t_0 = t$. Also, note that $a_x = a_{0x}$ (constant acceleration). With all this, we now state the kinematical equations.

$$v_x = v_{0x} + a_x t, \tag{2.6}$$

$$(x - x_0) = v_{0x} t + \frac{1}{2} a_x t^2, \tag{2.7}$$

$$v_x^2 = v_{0x}^2 + 2a_x(x - x_0). \tag{2.8}$$

If a particle is undergoing motion only along the y-axis, then the above equations are valid with the label replacement $x \to y$.

The first of the above three equations is a result of integrating the definition $v_x = dx/dt$. The second is got by integrating the first equation further. And the third equation can be had from the first and the second by some algebraic manipulation, which eliminates time t.

2.5.1 Free fall

An object is said to be in free fall if the only force acting on it is the gravitational force. Unless stated otherwise, the context of the Earth will be assumed. In free fall we have $\vec{a} = -g\hat{j}$, where $g \sim 9.8 m/s^2$ close to the surface of the Earth, and $\hat{j}$ points upwards (away from the center of the Earth). The reason why $\vec{a}$ has this value will be revealed in the Chapter on Newton's laws. g is called the "acceleration due to gravity" (of a certain celestial body). However it is a misnomer because g is actually the magnitude of the acceleration of an object in free fall.

2.6 Problem solving tips and strategies for this course

- Set up the coordinate axes (frame). Many physics problems are independent of the choice of the frame; so we choose the most convenient one.

- Draw a diagram relevant to the situation, showing your coordinate axes.

- Identify the correct equations to use.

- Don't plug in numbers just yet. Solve in symbolic form to the greatest degree as possible. Only in the final stage plug in the numbers if you have to. Symbolic answers let us perform some sanity checks.

- With the answer present in the symbolic form, perform sanity checks. (i) Does the answer have the right sign? (ii) See what does your symbolic answer reduce to in some particular limit, and if it makes sense or not. (iii) Does the answer have the right units?

- Before plugging in numbers, state clearly the numerical values of the symbols that you are about to plug in.

- After plugging in the numbers, see if the numerical answer is sensible or not (a human can't have a mass of 4000 kg).

2.7 Exercises

1. Starting from the origin, you 200 m in the x direction in 4 s (segment A). Then you go 500 m in 15 s (segment B). Your velocity (vector) is constant in both the segments. Find the (i) net displacement (vector) (ii) distance traveled (iii) average velocity (iv) velocity during segment B (v) average speed (vi) speed during segment B[2].

2. The position of a car is given by $x(t) = at^3 - bt$. Assume $a = 3$, and $b = 2$, both in SI units. Find (i) these SI units of a and b assuming SI units throughout. (ii) the average velocity of the car from $t = 0$ s to $t = 10$ s. (iii) the acceleration at $t = 5$ s (iv) velocity when the acceleration is 10 m/s^2

3. Problem 31 from Chapter 3 of Ref. [2]

4. Problem 32 from Chapter 3 of Ref. [2]

5. (Taken from Ref. [1]) A 15-kg rock is dropped from rest on the earth and reaches the ground in 1.75 s. When it is dropped from the same height on Saturn's satellite Enceladus, it reaches the ground in 18.6 s. (i) What is the "acceleration due to gravity" (g_{Enc}) on Enceladus[3] (ii) Solve this problem using a frame whose vertical y-axes have been flipped.

6. Trains A and B are heading in the north direction (A is behind B) with speeds 100 m/s and 50 m/s, and accelerations 10 m/s^2, and 3 m/s^2, respectively. The initial distance between them is 100 m. When and where do the trains collide?

7. You are aiming to drop an egg on a person's head, who is 2 m tall from a building of height of 50 m. The person is walking at a speed of 1.5 m/s alongside that building. How far does the person have to be (horizontally) from the spot of drop so that you don't miss the target.

8. You fire a bullet with a speed of 1000 m/s vertically up in the air. (i) With what speed is it going to crash on your head on its way down? (ii) Solve this problem using a frame whose vertical y-axes have been flipped.

9. A ball is thrown vertically up in the air with a speed 10 m/s. (i) What is its speed at the highest point from the point of launch? (Hint: use maxima/minima calculus). (ii) How high is this highest point? (iii) When does the ball attain zero speed?

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[2]Note that average speed is not the magnitude of the average velocity, although speed is defined to be the magnitude of velocity. This problem will help you see so.

[3]This also gives us a nifty way to calculate M/R^2 (M = mass, R =radius) of a celestial body, for $g = GM/R^2$, where G is Gravitational constant. We will learn this fact in the chapter on Newton's laws.

Chapter 3

Kinematics in two dimensions

3.1 Basic kinematical definitions

Calculus tidbit:

$$\frac{d(Cf)}{dt} = C\frac{df}{dt},\tag{3.1}$$

where C is a constant scalar or even a *constant vector*.

The position vector with components (x, y) is written as

$$\vec{r} = x\hat{i} + y\hat{j}.\tag{3.2}$$

The displacement vector (corresponding to initial and final positions $\vec{r}_i$ and $\vec{r}_f$) is

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i = (x_f - x_i)\hat{i} + (y_f - y_i)\hat{j}.\tag{3.3}$$

Average velocity vector between two time instants (which are Δt apart) is

$$\vec{v}_{\text{av}} = \frac{\Delta\vec{r}}{\Delta t}.\tag{3.4}$$

(Instantaneous) velocity vector is

$$\vec{v} = \lim_{\Delta t \to 0}\frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = v_x\hat{i} + v_y\hat{j}.\tag{3.5}$$

The speed is defined as $v = |\vec{v}|$.

It follows from the above definition that $\vec{v}$ is tangent to the trajectory. For the special case when the trajectory is a straight line, then this statement turns into the one stating that velocity points in the same direction of motion.

Average acceleration vector between two time instants (which are Δt apart) is

$$\vec{a}_{\text{av}} = \frac{\Delta\vec{v}}{\Delta t}.\tag{3.6}$$

(Instantaneous) Acceleration vector is

$$\vec{a} = \lim_{\Delta t \to 0}\frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} = a_x\hat{i} + a_y\hat{j}.\tag{3.7}$$

This above definition leads to some interesting consequences for a particle undergoing a motion on a curved trajectory at a fixed speed, i.e. it has a non-zero acceleration and that its acceleration points towards the concave side of the trajectory. (show pictorially). This makes us surmise that non-tangential (perpendicular to $\vec{v}$) component of $\vec{a}$ is responsible for changing the direction of $\vec{v}$. This statement needs to be made more precise.

We can decompose $\vec{a}$ into a parallel and a perpendicular (to $\vec{v}$) component.

$$\frac{dv}{dt} = \frac{(v_x^2 + v_y^2)^{1/2}}{dt}, \tag{3.8}$$

$$= \frac{v_x a_x + v_y a_y}{v}, \tag{3.9}$$

$$= \hat{v} \cdot \vec{a}, \tag{3.10}$$

$$= \text{component of acceleration parallel to the velocity.} \tag{3.11}$$

which makes us conclude that $dv/dt = 0$ if $\vec{v} \perp \vec{a}$. In other words, no parallel component of acceleration implies no change in speed. Summarizing, perpendicular and parallel (to $\vec{v}$) components of $\vec{a}$ are responsible for changing the direction and magnitude (speed) of the velocity, respectively.

3.2 Projectile motion

3.2.1 Kinematical equations

For projectile motion, we have the following two sets of equations valid simultaneously for each of the two dimensions

$$v_x = v_{0x} + a_x t, \tag{3.12}$$

$$(x - x_0) = v_{0x} t + \frac{1}{2} a_x t^2, \tag{3.13}$$

$$v_x^2 = v_{0x}^2 + 2 a_x (x - x_0). \tag{3.14}$$

$$v_y = v_{0y} + a_y t, \tag{3.15}$$

$$(y - y_0) = v_{0y} t + \frac{1}{2} a_y t^2, \tag{3.16}$$

$$v_y^2 = v_{0y}^2 + 2 a_y (y - y_0). \tag{3.17}$$

For the special case where we place the origin of our frame at the launch point, with y-axis pointing upwards, we have $a_x = 0 \text{ m/s}^2$, $a_y = -g$, $x_0 = y_0 = 0$ m, thus simplifying the above equations to

$$v_x = v_{0x}, \tag{3.18}$$

$$x = v_{0x} t, \tag{3.19}$$

$$v_y = v_{0y} - gt, \tag{3.20}$$

$$y = v_{0y} t - \frac{1}{2} g t^2, \tag{3.21}$$

$$v_y^2 = v_{0y}^2 - 2gy. \tag{3.22}$$

3.2.2 Maximum height, horizontal range, and time of flight

Here we specialize ourselves to the case where the initial and final points are at the same height. The point of maximum height is characterized by $v_y = 0$. To see why, apply the calculus concepts

that maximum or minimum with derivatives. Using $v_y = 0$ in the above equations of Sec. 3.2.1 lets us arrive at the maximum height, which turns out to be

$$H = \frac{v_0^2 \sin^2 \theta}{2g}. \tag{3.23}$$

The time of flight is

$$T = \frac{2v_0 \sin \theta}{g}. \tag{3.24}$$

The horizontal range is

$$R = \frac{v_0^2 \sin 2\theta}{g}. \tag{3.25}$$

Here θ is the angle of launch. The θ that gives the maximum range (for a fixed v_0) can be computed by applying the calculus concept of maxima/minima. The angle is 45°.

Other symbols have usual meanings. I don't recommend memorizing these three results since they can be easily derived from the more fundamental kinematical equations, as we will do so in the class.

3.3 Motion in a circle

3.3.1 Uniform circular motion

When an angle or its corresponding angular velocity is written symbolically (like θ, ϕ, ω), it will be assumed to be in radians, so that the familiar result $dr = d\theta \times R$ holds true (dr = small arc length; R = radius of the corresponding circle).[1] We already saw in Sec. 3.1, that $\vec{a}$ tends to point towards the concave side of a curved trajectory. Specializing to the case of a uniform (constant speed) circular motion, it's easy to see pictorially that the $\vec{a}$ points towards the center of the circle.

Now let's focus on a. Refer to Fig. 4.18 of Ref. [2]. Assume $\Delta\theta \to 0$, so that we can replace $\Delta \to d$ throughout. In the context of this figure,

$$|\Delta\vec{v}| = |d\vec{v}| = v d\theta, \tag{3.26}$$

$$a = |\vec{a}| = \left|\frac{d\vec{v}}{dt}\right|, \tag{3.27}$$

$$a = \frac{v d\theta}{dt} = \frac{2\pi v}{\Delta T} = \frac{2\pi v^2}{2\pi R}, \tag{3.28}$$

$$a = \frac{v^2}{R}. \tag{3.29}$$

Here $\Delta T = 2\pi R/v$ is the time taken to go around the circle. Only in radian measure, the relation $|d\vec{v}| = v d\theta$ holds true, which we used above.

Now $dr = d\theta \times R \implies v = \omega \times R$ ($v = dr/dt$, and $\omega = d\theta/dt$), we can rewrite Eq. (3.29) as

$$a = \omega^2 R. \tag{3.30}$$

3.3.2 The general case of non-uniform circular motion

Non-uniform circular motion refers to the general case when the body moves in a circle with varying speed. Let us set up some conventions first.

---

[1]The corresponding result $s = \theta \times R$ (s = arc length) is valid for larger angles too

The situation in consideration is that of a particle going around in a circle (of radius r) either only clockwise, or anti-clockwise, i.e. it never changes its sense of rotation. From a starting reference time $t = 0$, to a later time t it has swept an angle θ. $\theta > 0$ if it's going counterclockwise (CCW), and < 0 if it's going clockwise (CW). Also, at any point during the journey, set up a coordinate axis (call it the "circular frame") with its origin at the particle's position, so that its x-axis points towards its $\vec{v}$, and its y-axis points towards the center of the circle. Then we have the following relations (with s representing the distance traveled, and θ being the angular displacement)

$$s = \pm r\theta, \tag{3.31}$$

which upon differentiating gives (with $\omega \equiv d\theta/dt$ standing for angular velocity, and $|\omega|$ being the angular speed)

$$v = \frac{ds}{dt} = \pm r\frac{d\theta}{dt} = \pm r\omega. \tag{3.32}$$

We now state a result without proof. We a particle in circular motion,

$$\vec{a} = \frac{dv}{dt}\hat{i} + \frac{v^2}{r}\hat{j} \equiv a_t\hat{i} + a_r\hat{j}. \tag{3.33}$$

We use the $+$ $(-)$ sign in Eqs. 3.31 and 3.32 if the motion is CCW (CW). In Eq. 3.33, the two unit vectors are those associated with the so-called circular frame introduced above. A further differentiation of Eq. 3.32 leads to (with $\alpha \equiv d\omega/dt$ standing for angular acceleration)

$$\frac{dv}{dt} = a_t = \pm r\alpha. \tag{3.34}$$

The subscripts "r" and "t" in Eq. 3.33 refer to "radial" and "'tangential", respectively. We won't prove Eq. 3.33 fully. Note that the x-component of the result contaiend in Eq. 3.33 is essentially Eq. 3.11 (which we did derive), specialized to circular trajectories. The y-component of the result is well motivated from our derivation of equivalent result for uniform circular motion in Eq. 3.29.

3.4   Relative motion

In this section, we will talk about a particle P's kinematical quantities wrt. various frames. So, we invoke the following notation. We denote the position of point P in frame A as $\vec{r}_{PA}$. This convention applies to even velocity and acceleration. We will also assume that none of the frames is rotating wrt. another.

It's easy to see pictorially that

$$\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA}, \tag{3.35}$$

where $\vec{r}_{BA}$ stands for the position of the *origin* of Frame B wrt. Frame A. Differentiating twice successively yields

$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}, \tag{3.36}$$
$$\vec{a}_{PA} = \vec{a}_{PB} + \vec{a}_{BA}. \tag{3.37}$$

As far as the kinematical equations are concerned, none of the frames is too sacrosanct that it can't be abandoned, not even the one attached to the Earth. Working in a "moving frame" may simplify a problem.
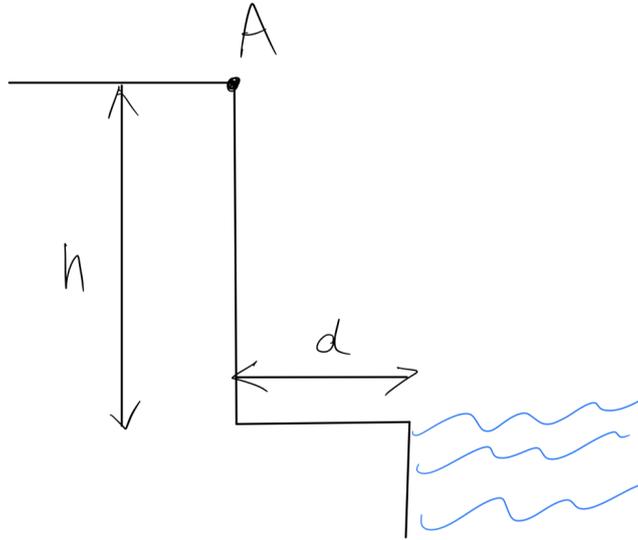
Figure 3.1:

3.5 Exercises

1. The position of a raccoon is $\vec{r}(t) = (3.0 \text{ m/s} \times t + 4.0 \text{ m/s}^2 \times t^2)\hat{i} - 2 \text{ m/s}^3 \times t^3 \hat{j}$. (i) What are the x and y-components of the velocity vector of the raccoon? (ii) At $t = 10$ s, how far is the raccoon from the initial position? (iii) What is its acceleration at $t = 10$ s?

2. In the context of Fig. 3.1, with what minimum speed should a person from the point A leap horizontally so that they land safely into the water?

3. A rock is thrown from the roof of a 10 m tall building at a launch angle of $60°$ at a speed of 30 m/s. Find (i) the horizontal range. (ii) the maximum height above the ground (iii) the speed when it hits the ground (iv) the x-t and y-t graph.

4. The crew aboard an airplane needs to deliver a package at a certain spot on the ground. The plane is heading in the direction of the spot at a height h and with speed v. (i) At what horizontal distance from that spot should the crew let go of the packet? (ii) When the packet is let go, what is the distance (not just horizontal distance) of the spot on the ground and the plane?

5. Humans lose consciousness when undergoing accelerations of magnitude more than $a_c = 6g$. A huge circular rotor is rotating with an angular velocity ω. How far from the center can you safely station yourself so that you don't lose consciousness?

6. Solve problem 6 (only the "time part") of Sec. 2.7 using the concept of relative velocities. Station the frame on the train that is behind the other train.

7. You are walking in the rain with a speed of v, whereas the rain is falling vertically with a speed v_r. At what angle you should tilt your umbrella to protect yourself from rain?

Chapter 4

Newton's laws

4.1 Newton's laws

4.1.1 Newton's second law:

$$\sum_i \vec{F}_i = m\vec{a}, \tag{4.1}$$

where $\sum_i \vec{F}_i$ is the net force on a certain body, m and $\vec{a}$ are its mass and acceleration. $\vec{a}$ in general depends on the frame of reference. $\vec{a}$ in Eq. (4.1) is to be measured in an inertial frame of reference. A frame attached to a body which far and distant from other material bodies can be considered to be an inertial frame of reference. For our purposes we will consider the Earth to be an inertial frame.

Note that

- It's the sum of forces of the RHS of Eq. (4.1), and not any individual force.

- The effect of a sum of forces is to impart a non-zero $\vec{a}$, rather than a fixed non-zero $\vec{v}$.

4.1.2 Newton's first law:

When the RHS of Eq. (4.1) is a zero vector, $\vec{a} = 0$ implying that $\vec{v}$ stays constant.

4.1.3 Newton's third law:

$$\vec{F}_{\text{A on B}} = -\vec{F}_{\text{B on A}}. \tag{4.2}$$

Note that the above two forces in Eq. (4.2) act on different bodies, and are applied by different bodies.

4.2 Mass and weight

Mass is an intrinsic property of a material body and signifies the body's resistance to being accelerated by forces; see Eq. (4.1). Weight $\vec{w}$ is the gravitational force that acts on a body. The magnitude of the weight is given by

$$w = |\vec{w}| = mg, \tag{4.3}$$

and it points towards the center of the Earth (or a certain celestial body). The unit of weight is kg m/s^2 = Newton (N). The unit of mass is kg. 1 kg = 2.2 lbs. In everyday language, when we use the term weight, we actually are talking about the mass.

Now in the chapter on gravitation, we will learn that the gravitational force on a body of mass m near the surface of the Earth is directed towards the center of the Earth, and has the magnitude

$$F_g = \frac{GMm}{R^2}, \tag{4.4}$$

where G = gravitational constant, R = radius of the Earth, and M is the mass of the Earth. Now, Newton's 2nd law immediately gives us the magnitude of the acceleration of this body in free fall as

$$a = \frac{F_g}{m} = \frac{GM}{R^2} = g \sim 9.8 \text{ m/s}^2, \tag{4.5}$$

a fact that we simply assumed in the last two chapters.

4.3   Some common forces

- **Force of gravity (mg):** Every object near the surface of the Earth is being acted upon by the force of gravity of magnitude mg. Its direction is towards the center of the Earth.

- **Tension (T):** A taut string applies a pulling force at both of its ends. This is tension.

- **Normal (N):** A rigid body may apply a pushing force perpendicular to its surface. This is called normal. If the surface is curved (a sphere), draw a tangent to the surface through a certain point. Then the normal at that point points in a direction that is perpendicular to the surface. A weighing scale reads the normal force (between the scale and the body whose weight is to be measured). The mass it output is N/g kg.

- **Friction (f):** A rigid body A applies a force parallel to its surface on a body B which is in contact with A to prevent relative motion between A and B. This is called the force of friction.

4.4   Friction

Friction between two bodies A and B, which are in contact acts parallel to the surface of contact to prevent any relative motion between A and B. It falls in two categories (i) static friction (ii) kinetic friction

Static friction acts when their is no relative motion between A and B.

$$f_s \leq \mu_s N. \tag{4.6}$$

μ_s is the coefficient of static friction and depends on the nature of material of the two bodies A and B. Kinetic friction acts when there is relative motion (slippage) between A and B.

$$f_k = \mu_k N. \tag{4.7}$$

μ_k is the coefficient of kinetic friction and again depends on the nature of material of the two bodies A and B.

Because $\mu_s > \mu_k$, once the block has overcome static friction (where the static friction = applied force), the block actually moves with an acceleration thereafter (because kinetic friction < applied force). Explain with a diagram.
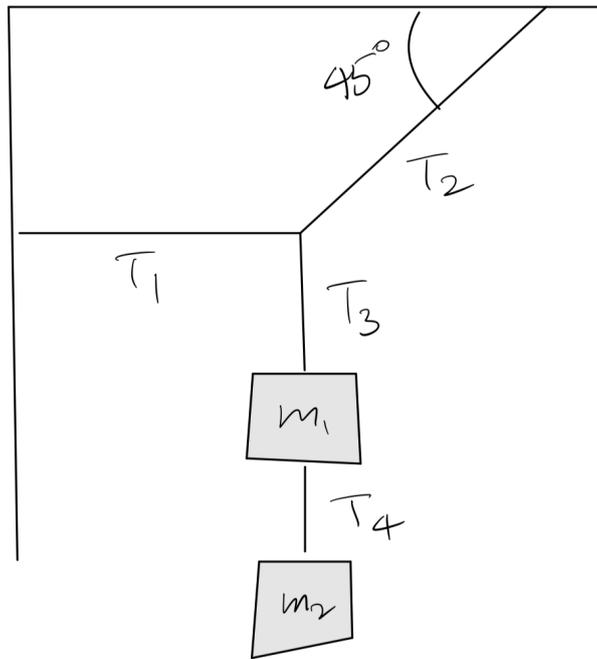
Figure 4.1:

4.5 Free-body diagrams

1. First a particular Body A is singled out for which a free-body diagram (FBD) is sought.

2. Include only the *external*[1] forces that act on that on Body A. Don't include the forces that Body A applies on other bodies.

3. Usually, you should be able to answer which body applies a particular force on Body A.

4. Choose a convenient coordinate system. Decompose all the forces along its x and y-axes.[2]

5. Apply Newton's 2nd law ($\sum_i \vec{F}_i = m\vec{a}$) along these axes, which with the help of the rules given in Sec. 1.4) reads

$$\sum_i F_{xi} = ma_x, \tag{4.8}$$

$$\sum_i F_{yi} = ma_y. \tag{4.9}$$

4.6 Exercises

Unless stated otherwise, ropes, strings, and pulleys are to be considered massless, and surfaces frictionless.

1. A block of mass m_b is suspended from the ceiling with a rope of length l. (i) Determine the tension in the rope. (ii) Now assume the rope has a (uniformly distributed) mass m_r. Find the tension in the rope at a point which is at a distance d from the ceiling.

---

[1]A force that is applied on the system by a body within the sytem is said to be internal. The forces which are not internal are called external forces.

[2]Force physically "add" as per addition law of vectors. This justifies replacing a force with its x and y components (a very useful technique.
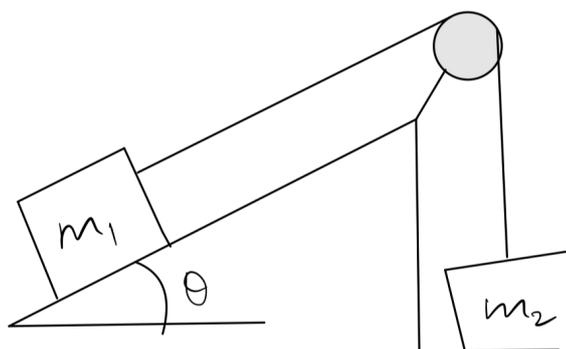
Figure 4.2:

2. For the system shown in Fig. 4.1, determine the tensions in all the strings.

3. For the system shown in Fig. 4.2, (i) determine the acceleration of the two bodies. (ii) What condition must be satisfied by the two masses for there to be no acceleration?

4. An elevator of mass M moves downward with constant speed. (i) Determine the tension in the cables. Then the elevator starts to slow down (decelerate). (ii) What is the direction of acceleration vector during this stage? (iii) Assuming the maximum tension that cable can handle is T_{max}, determine the fastest rate (acceleration magnitude) at which the elevator can safely decelerate. (iv) Now imagine a person of mass m (standing on a very light weighing scale) is inside the elevator while its decelerating at rate a (magnitude of the acceleration vector). What mass does the weighing scale read?

5. An inclined plane (with inclination θ) and a block (of mass m) combo have μ_s and μ_k as their two coefficients of friction. Assume no friction ($\mu_s = \mu_k = 0$) only for part (i) below. (i) Determine the acceleration of the block at a certain inclination θ. (ii) At what critical inclination θ_c does the block start to slide? (iii) Determine the acceleration when the block has started to slide at an inclination inclination θ, which is larger than θ_c.

6. A wire goes over a pulley and is connected to blocks of masses m_1 and m_2 on either end. Determine the acceleration of the two blocks.

7. You are pulling a big crate of mass m with a rope that goes over your shoulder and is inclined wrt. the horizontal at an angle of θ. The coefficients of friction are μ_s and μ_k for the ground-crate combo. (i) What is the minimum tension need to pull the crate? (ii) With this tension, what is the acceleration of the crate?

8. You are sitting in a sled that is connected with a rope to a central post. Assuming your neck can withstand a maximum force of F_{max}, what is the maximum safe ω with which the sled can go around the post? Assume that the mass of your head is m.

9. A conical pendulum sweeps a cone around the vertical line that goes through the point where the pendulum is attached to the ceiling. The pendulum is composed of a wire of length l, a bob of mass m, and makes and angle θ with the vertical. Find (i) the tension in the wire and (ii) time period of one revolution.

10. Determine the angle at which a frictionless curved street (of radius R) should be banked so that a car moving with a certain speed v can turn.

11. Two boxes of masses m_1 and m_2 are connected with a string. They are placed on a surface. The first box (of mass m_1) is pulled with force F_p which results in the second box being dragged too with a common acceleration of a. The friction coefficient is μ_k. (i) Determine F_p (ii) Determine the tension in the string.

12. A fly smashes against a vertical windshield of a car and dies. With what acceleration should the car move so that the fly does not slide down? Assume the coefficient of friction to be μ.

Chapter 5

Work and energy

5.1   Integral calculus review

If $dg(x)/dx = f(x)$, then $g(x)$ is called the integral or anti-derivative of $f(x)$. Symbolically,

$$\int f(x)dx = g(x). \tag{5.1}$$

With n being an integer

$$\int x^n dx = \frac{x^{n+1}}{n+1}. \tag{5.2}$$

Also, if $\int f(x)dx = g(x)$, then

$$\int_a^b f(x)dx \equiv \lim_{\Delta x \to 0} \sum_i f(x_i)\Delta x = g\big|_{x=b} - g\big|_{x=a} \equiv g(b) - g(a) = \Delta g, \tag{5.3}$$

Applying these rules to $\int_{x_i}^{x_f} dx$ gives

$$\int_{x_i}^{x_f} dx = \int_{x_i}^{x_f} x^0 dx = x_f - x_i = \Delta x, \tag{5.4}$$

because when $f(x) = x^0 = 1$, $g(x) = \int f(x)dx = x$ from Eq. 5.2. The geometric interpretation of $\int_a^b f(x)dx$ is that it is the area under the curve representing $f(x)$ between $x = a$, and $x = b$, where the differential area $f(x)dx$ is positive if $f(x) > 0$, and vice versa. We also have

$$\int C f(x)dx = C \int f(x)dx, \tag{5.5}$$

where C is a constant scalar or a vector.

5.2   Work-energy theorem

Work is defined (from point P to Q connected with a trajectory) as

$$W = \int_P^Q \vec{F} \cdot d\vec{r}, \tag{5.6}$$

which we can further process as

$$W = m \int_P^Q \frac{d\vec{v}}{dt} \cdot d\vec{r} = m \int_P^Q d\vec{v} \cdot \frac{d\vec{r}}{dt} = m \int_P^Q \vec{v} \cdot d\vec{v} = \frac{m}{2}\left(v^2\big|_{v_Q} - v^2\big|_{v_P}\right) = \frac{m}{2}\left(v_Q^2 - v_P^2\right) \tag{5.7}$$

$$= \Delta\left(\frac{1}{2}mv^2\right) = \Delta\text{KE}, \tag{5.8}$$
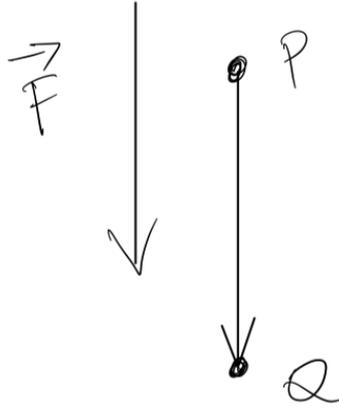
Figure 5.1:

where just defined the kinetic energy KE to be

$$\text{KE} = \frac{1}{2}mv^2. \tag{5.9}$$

Eq. (5.8) ($W = \Delta KE$) is the work-energy theorem.

The work done by a constant force becomes

$$W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r} = \vec{F} \cdot \int_{\vec{r}_i}^{\vec{r}_f} d\vec{r} = \vec{F} \cdot \Delta\vec{r}, \tag{5.10}$$

where we used Eq. 5.5 in the second equality, and Eq. 5.4 in the third one.

We can also quantify the rate at which work is done by defining power as

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}. \tag{5.11}$$

5.3 Potential energy and mechanical energy conservation

5.3.1 Potential energy (one dimension)

We confine ourselves to one dimension only. Consider a force $\vec{F}$ whose magnitude depends on the position, and possibly some other constants. Such forces are called conservative forces. Examples of conservative forces are the force of gravity ($F = Gm_1m_2/r^2$) and spring force ($F = kd$). Non-conservative forces are friction, normal, tension and forces of collision. Refer to Fig. 5.1 A particle is let go from P to Q under the effect of only the conservative force $\vec{F}$, or rather $F_x(x)$ (if the x-axis points in the direction of $\vec{F}$). The work done is (from Eq. 5.3)

$$W_{P\to Q} = \int_P^Q F_x(x)dx = G(Q) - G(P) = \Delta G, \tag{5.12}$$

where

$$G = \int F_x(x)dx. \tag{5.13}$$

Note that

$$\Delta G = \Delta\text{KE}, \tag{5.14}$$

since the changes in both G or KE are obtained by $\int_P^Q \vec{F} \cdot d\vec{r}$.

This now implies that if we somehow slowly drag the particle back from Q to P, we somehow "store a potential" in that particle to gain a kinetic energy of $G(Q) - G(P)$ (if it is let go in the same manner again). This potential storage/increase is quantified by inventing the term potential energy (PE), and we say that the particle's PE has increased by $G(Q) - G(P)$. Mathematically (with $H \equiv -G$),

$$\Delta\text{PE}_{Q \to P} = G(Q) - G(P) = H(P) - H(Q) = H_{\text{final}} - H_{\text{initial}}, \tag{5.15}$$

which motivates us to define the PE as

$$\text{PE} = H. \tag{5.16}$$

Finally, we note that

$$W_{P \to Q} = \Delta G = -\Delta H = -\Delta\text{PE}, \tag{5.17}$$

a fact we will make use of later.

5.3.2   Conservation of mechanical energy (one dimension)

Mechanical energy ME is

$$\text{ME} = \text{KE} + \text{PE}. \tag{5.18}$$

Imagine a body being acted upon only by conservative forces, and possibly by forces which do no work (see Sec. 5.3.4). Under this condition,

$$\Delta\text{ME} = \Delta\text{KE} + \Delta\text{PE} = \Delta G + \Delta H = 0, \tag{5.19}$$

since $G = -H \implies \Delta G = -\Delta H$, thereby showing that the mechanical energy is conserved[1]

5.3.3   Conservation of mechanical energy in more than one dimension

We state without proof that the above principle of conservation of mechanical energy holds even in more than one dimension.

5.3.4   Work done by non-conservative forces

Total work done by the non-conservative forces of tension is 0. This can play out in two separate ways: (i) tension does no work, as in the case of pendulum bob, where $\vec{F}_T \perp d\vec{r}$, thus giving $\vec{F}_T \cdot d\vec{r} = 0$ (ii) the individual pieces of work done by tension cancel each other out, as happens with a string (connected with masses) going over a pulley.

The total work done by the non-conservative normal force is also zero in a similar way. (i) it may be so that all individual pieces of work done by normal is 0 (because displacement is perpendicular to normal, thereby giving $\vec{F}_n \cdot \Delta r = 0$), as happens when a block is sliding on a table. (ii) individual pieces of work done by normal forces are not 0, but they cancel each other out. An example of the latter is a person in an elevator. The action-reaction normal pair between the person and the elevator do work which are negatives of each other and hence give 0, when added together.

So, there is a complete guarantee that only the total work done by normal and tension forces is 0. This guarantee does not hold for the individual pieces of work done by them.

Finally, a word about friction. The total work done due to static friction is zero, again due to two separate mechanisms: (i) when their is no motion (you trying to push a very heavy box), the static friction does no work (ii) when a static friction does work, its corresponding action-reaction static friction does a work of equal magnitude but with opposite sign, thereby giving zero total work.[2] Kinetic friction, however does negative total work.

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[1]Note that despite the fact that conservation of energy is touted as a law, we have been able to derive it from Newton's laws.

[2]This means that no slippage implies zero total work done by friction.

5.3.5 General scenario: non-conservation of mechanical energy with non-conservative forces

Consider the most general scenario where both conservative and non-conservative forces are acting simultaneously. In this case,

$$\Delta\text{ME} = \Delta\text{KE} + \Delta\text{PE} = (W_{\text{Cons}} + W_{\text{Non-cons}}) + (-W_{\text{Cons}}) = W_{\text{Non-cons}}, \tag{5.20}$$

where we used Eq. 5.17 in the second equality above. Note that Eq. 5.19 is a special case of Eq. 5.20 when there there are no non-conservative forces.

With a clever choice of the system in consideration, in many instances, we can reduce $W_{\text{Non-cons}}$ in Eq. 5.20 to just W_{fric} (work done by friction), because we saw in Sec. 5.3.4, that the total work done by the non-conservative forces of normal and tension is 0.

5.4 Potential energy for specific forces

5.4.1 Spring force

Spring (of spring constant k) when compressed by d applies a pushing force of magnitude kd, and when stretched by d applies a pulling force of magnitude kd. In short, $F = kd$.

With the x-axis pointing from the anchor of the spring to the connected body, and the origin ($x = 0$) at the open end of the spring, when the spring is in its neutral state, we have

$$F_x = -kx, \tag{5.21}$$

$$\implies G = \int F_x dx = -\frac{1}{2}kx^2, \tag{5.22}$$

$$\implies \text{PE} = H = -G = \frac{1}{2}kx^2. \tag{5.23}$$

In other words,

$$\text{PE} = \frac{1}{2}kd^2. \tag{5.24}$$

5.4.2 Gravitational force

With the y-axis pointing upwards, we have

$$\vec{F} = -mg\hat{j}, \tag{5.25}$$

$$\implies G = \int \vec{F} \cdot d\vec{r}, \tag{5.26}$$

where the above equation is a 2D generalization of Eq. 5.13. Proceeding further,

$$G = \int \vec{F} \cdot d\vec{r} = \int -mg\hat{j} \cdot (dx\hat{i} + dy\hat{j}) = \int -mgdy = -mgy, \tag{5.27}$$

$$\implies \text{PE} = H = -G = mgy. \tag{5.28}$$

In many references, PE in the context of gravity is referred to as mgh, where h is the height above ground.

Remark: We can easily solve mechanical problems using energy conservation methods, which would otherwise be cumbersome to impossible to solve using the older kinematical equations or Newton's laws. The presence of unknown normal forces in many situations presents no problem when the energy-conservation methods are applied to solve the system, the reason being that the work done by normal is zero. We will see this when we do the exercise problems.
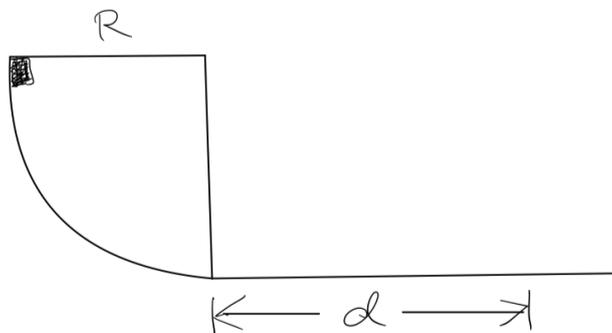
Figure 5.2:

5.5 Exercises

All surfaces are assumed to be frictionless unless stated otherwise. All ropes, strings and pulleys are assumed to be light unless stated otherwise.

1. A box of mass m is pulled by a rope (with tension T, inclined by an angle θ wrt. the horizontal) from a state of rest over a surface with coefficient of friction μ_k over a distance d. (i) Find the work done by each force (ii) Find the final speed.

2. A ball is thrown up in the air with a speed v. (i) How high does it go up (using kinematical equations)? (ii) Solve the above problem using energy conservation.

3. On a table, a block of mass m is compressed (by distance d) against a spring of spring constant k. Upon released, the block travels a total distance of D (measured from the initial point of compression) before coming to rest. Find the coefficient of friction between the block and the table.

4. On a table, a block of mass m is compressed (by distance d) against a spring of spring constant k. Upon released, the block travels horizontally for a while before climbing up an incline with inclination θ. How far along the incline does the block go? Assume that the incline begins after the block travels a distance d.

5. Two blocks of masses m_1 and m_2 ($m_1 > m_2$) are connected with a rope which passes over a pulley. Blocks 1 and 2 are at a height h and 0 m above the ground, respectively. The system is then let go from rest. (i) Find the speed with which Block 1 strike the ground using Newton's laws. (ii) Do the same problem using energy conservation.

6. A particle of mass m starts to slide from the top of a hemispherical bowl. At what angle from the vertical does the particle lose contact with the bowl?

7. A block of mass m slides from the top (from a state of rest) of the track in Fig. 5.2 and covers a distance d on the horizontal section of the track. Only the horizontal section of the track is rough; the rest is smooth. (i) What is the coefficient of friction for the horizontal section? (ii) Find the work done by friction.

8. A block and a spring (of mass m and spring constant k, respectively) are on a table, which is at a height h above the ground. The block is held against the compressed spring (by distance d), and then let go. What is the speed of the block when it lands on the floor?

9. A basket of mass m_1 hangs from the ceiling through a spring of spring constant k. A block of mass m is gently placed into the basket. How much further does the spring stretch?

10. A block of mass m goes around a circular hoop of radius R. At a certain instant, at its bottommost point, the normal is N_1, then sometime later the block is at the topmost point, where the normal is N_2. Find the work done by friction between these two instances.

11. A pendulum of mass m, and length l swings with a maximum angle $\theta = \theta_m$. (i) Find the speed of the bob at the lowest point. (ii) What is the tension at the lowest point $\theta = 0$? (iii) What is the tension at the highest point $\theta = \theta_m$?

12. Redo Problem 5 (ii) by applying Eq. 5.20 only to Block 1. Note that in this case, we cannot assume the work due to tension to be equal to 0 (as you may have done while doing Prob. 5 if you considered the composite system Block 1 + Block 2). Think why. Discussion in Sec. 5.3.4 might help.

Chapter 6

Collisions

6.1 Impulse-momentum theorem

Newton's 2nd law says that

$$\sum \vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt}, \tag{6.1}$$

$$\Delta \vec{p} = \int \sum \vec{F} dt \equiv \vec{J}. \tag{6.2}$$

where we have defined the momentum $\vec{p} \equiv m\vec{v}$. The impulse $\vec{J}$ is the RHS of the above equation. Impulse of a certain force is a vector and is defined as the time definite integral of that force vector. Therefore during a collision (interactions lasting for a very short duration), appreciable momentum change is brought about forces which are quite high in magnitude. Hence, during a collision, forces of gravity, friction etc become irrelevant since their magnitudes are not big (unlike collision forces). Forces of collision matter the most. They can also be non-conservative in general, which implies that they need not conserve energy.

6.2 Newton's 2nd law applied to a system of particles

6.2.1 Center of mass

Center of mass for a system of particles with masses $m_1, m_2, m_3, \ldots$ is defined as

$$\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i}{M}, \tag{6.3}$$

where $M \equiv \sum_i m_i$.

$$M\vec{r}_{cm} = \sum_i m_i \vec{r}_i, \tag{6.4}$$

$$\implies M\vec{v}_{cm} = \sum_i m_i \vec{v}_i = \vec{P}, \tag{6.5}$$

$$\implies M\vec{a}_{cm} = \sum_i m_i \vec{a}_i, \tag{6.6}$$

where $\vec{P}$ is the total momentum of the system.

6.2.2 Newton's 2nd law and conservation of momentum

$$\sum \vec{F} = \sum_i m_i \vec{a}_i \tag{6.7}$$

$$\sum \vec{F} = \sum \left(\vec{F}_{int} + \vec{F}_{ext} \right) = \sum \vec{F}_{ext} = \sum_i m_i \vec{a}_i = M\vec{a}_{cm} = \sum_i m_i \frac{d\vec{v}_i}{dt} = \frac{d(\sum_i m_i \vec{v}_i)}{dt} = \frac{d\vec{P}}{dt}, \tag{6.8}$$

$$\sum \vec{F}_{ext} = M\vec{a}_{cm} = \frac{d\vec{P}}{dt}, \tag{6.9}$$

which means that if the external forces add up to 0, then the total momentum of the system does not change (law of conservation of momentum). Furthermore,

$$\Delta \vec{P} = \int \sum \vec{F}_{ext} dt. \tag{6.10}$$

6.2.3 Energy conservation and non-conservation in collisions

Elastic collisions: Total initial kinetic energy (before collision) is the same as total final kinetic energy (after collision).
Inelastic collisions: Total initial kinetic energy (before collision) is larger than the total final kinetic energy (after collision).
Completely inelastic collisions: A kind of inelastic collision, where the bodies stick together after collision.

6.3 Exercises

1. A bullet of mass m fired horizontally with speed v gets lodged into a wooden block of mass M that is suspended from the ceiling with a string of length l. After getting lodged, the block and the bullet swing upwards by an angle θ. Determine v.

2. A ball of mass m is initially moving in the $-x$ direction with speed v. After being hit, it moves in the direction which is at an angle θ wrt. the x axis (in the first quadrant) with the speed $2v$. Find the total impulse imparted to the ball.

3. A rifle of mass M shoots a bullet of mass m with speed v. (i) What is the recoil velocity of the rifle? (ii) Find the final momentum vector of the bullet and the rifle.

4. Two cars of masses m_1 and m_2 head towards each other with speeds v_1 and v_2 from the left and the right directions, respectively. After collision, the second car has a speed v_3 to the right. Find the velocity vector of the first car. Hint: as discussed in the class, we can ignore friction.

5. Solve the above problem assuming that after the collision, the cars stick together. What percentage of the kinetic energy is lost? This is a completely inelastic collision.

6. Consider a 2D plane with coordinate axes installed. Car A of mass m_1 moving with a velocity $v_{1A}\hat{i}$ ($v_{1A} > 0$) collides with Car B (of mass m_2), which is initially at rest. After collision, Car A moves with a speed v_{2A} which makes an angle θ wrt. the x-axis (in the first quadrant). What is the final velocity vector of Car B?

Chapter 7

Rotation

When we write any angle or any of its derivatives (angular velocity or angular acceleration), we measure angles in radians. Also,

$$x° = x\frac{\pi}{180}\text{radians.} \tag{7.1}$$

7.1 Kinematics

7.1.1 Kinematical definitions

We import all the rotational kinematical definitions, conventions from Sec. 3.3.2. The reader is expected to be familiar with this section.

7.1.2 Kinematical equations

In parallel to the standard linear kinematical equations, we have in the rotational sector

$$\omega = \omega_0 + \alpha t, \tag{7.2}$$

$$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2, \tag{7.3}$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0). \tag{7.4}$$

These are results of the definitions $\omega = d\theta/dt$ and $\alpha = d\omega/dt$ that were introduced in Sec. 3.3.2.

7.2 Moment of inertia

7.2.1 Energy and moment of inertia

The kinetic energy of a rigid body rotating around a stationary z-axis is

$$K = \sum \frac{1}{2}m_i v_i^2 = \sum \frac{1}{2}m_i \omega^2 r_i^2 = \frac{1}{2}I_z \omega^2, \tag{7.5}$$

where

$$I_z = \sum_i m_i r_i^2, \tag{7.6}$$

is called the moment of inertia of the body (about the z-axis). The above expression for the kinetic energy is the rotational analog of $\text{KE} = (1/2)mv^2$, and I_z is the rotational analog of mass m.

7.2.2 Parallel axis theorem

One can show that I_P, the moment of inertia (MI) about an axis A, which is parallel to an axis B about the center of mass (CM) is

$$I_P = I_{cm} + Md^2, \tag{7.7}$$

where I_{cm} is the MI about the axis B, and d is the distance between the two axes.

7.2.3 Moment of inertia with calculus

It can be shown using integral calculus that I for a uniform rod (of length l) and a cylinder (of radius r) wrt. an axis through their centers are $(1/12)ml^2$ and $(1/2)mr^2$, where m represents the masses of either the rod or the disc. I for a sphere wrt. an axis through its center is $(2/5)mr^2$. I for a uniform rod wrt. its end point is $(1/3)ml^2$.

7.3 Dynamics

7.3.1 Cross product

Cross product between two vectors $\vec{A}$ and $\vec{B}$ is denoted as

$$\vec{C} = \vec{A} \times \vec{B}. \tag{7.8}$$

It is a vector whose magnitude is $C = AB\sin\phi$, where $0° \leq \phi \leq 180°$, and its direction is given by the right-hand rule: curl your fingers from $\vec{A}$ towards $\vec{B}$; your thumb points in the direction of $\vec{C}$.

7.3.2 Gravity

We state without proof that the force of gravity on a rigid body can be assumed to act at its center of mass. Also, its gravitational potential energy can be shown to be equal to mgh, where m is the mass of the body, and h is the height of its center of mass.

7.3.3 Torque and angular momentum

Torque due to a force $\vec{F}$ applied on a particle P, which has a position vector $\vec{r}$ wrt. the origin O of a coordinate system is

$$\vec{\tau} = \vec{r} \times \vec{F}. \tag{7.9}$$

This means that τ is given by the product of F and the moment arm of the force wrt. the origin, the moment arm being defined as the length of the perpendicular drawn from the origin O to the point of application of the force $\vec{F}$.

Torque can further be written as

$$\vec{\tau} = \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + m\vec{v} \times \vec{v} = \vec{r} \times \frac{d\vec{p}}{dt} + \vec{p} \times \frac{d\vec{r}}{dt} = \frac{d(\vec{r} \times \vec{p})}{dt} = \frac{d\vec{L}}{dt}, \tag{7.10}$$

where we have defined the angular momentum of the particle as $\vec{L} = \vec{r} \times \vec{p}$, and used the vector analog of the derivative product rule $d(fg)/dx = f\,dg/dx + g\,df/dx$. On a collection of particles, Eq. 7.10 easily generalizes to

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}, \tag{7.11}$$

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where $\vec{L}$ is the angular momentum of the entire system of particles, and on the LHS, we only have the torques of the external forces since those due to internal forces cancel each other out because of Newton's 3rd law. Eq. 7.11 is the rotational analog of $\sum \vec{F} = d\vec{p}/dt$.

Now consider a rigid body rotating about a fixed axis (z-axis) with an angular velocity ω. Consider a particle P of mass m_i of this body which is at a distance ρ_i from the z-axis. The z-component of the body's angular momentum (wrt. a point on the z-axis) can easily be shown to be

$$L_{zi} = m_i \rho_i^2 \omega, \tag{7.12}$$

which renders the z-component of the entire body's angular momentum to be

$$L_z = \sum L_{zi} = \omega \sum m_i \rho_i^2 = I_z \omega, \tag{7.13}$$

where I_z is the moment of inertia about the z-axis. Note that since $\omega < 0$ implies a CW rotation, $L_z < 0$ for such a rotation, and vice versa. Applying the z-component of Eq. 7.11

$$\sum \tau_{ext-z} = \frac{dL_z}{dt}, \tag{7.14}$$

to Eq. 7.13 gives us

$$\sum \tau_{ext-z} = I_z \alpha, \tag{7.15}$$

which is the rotational analog of Newton's 2nd law $\sum \vec{F} = m\vec{a}$. In Eqs. 7.14 and 7.15, one does not have to include the external force that the pivot point applies on the rotating body since the z-component of its torque about a point on the z-axis is zero.

7.3.4   Conservation of angular momentum

When there is no external torque acting on a system of bodies, Eq. 7.11 reduces to

$$\frac{d\vec{L}}{dt} = 0, \tag{7.16}$$

$$\vec{L} = \text{constant}, \tag{7.17}$$

$$L_z = I_z \omega = \text{constant (for a rigid body rotating about the z-axis)}, \tag{7.18}$$

thus leading to the conservation of angular momentum, or that of its component along the rotation z-axis.

7.4   A translating and rotating rigid body

7.4.1   Kinetic energy

This section is about a rigid body which translates and rotates such that the rotation axis always points in the same direction. We state without proof that its kinetic energy is

$$\text{KE} = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2. \tag{7.19}$$

Here M = body's mass, v_{cm} = body's center-of-mass (CM) speed, I_{cm} = body's moment of inertia about the CM, and ω = body's angular velocity. The interested reader is referred to Ref. [1] for a proof of this statement.

7.4.2 Rolling without slipping

Rolling without slipping means that the point of the body in contact with the ground is at rest wrt. the ground. This further implies that

$$v_{cm} = r|\omega|, \tag{7.20}$$
$$a_{cm} = r|\alpha|. \tag{7.21}$$

where r is the radius of the body.

7.4.3 Analog of Newton's 2nd law

From Eq. 6.9 we have the following equation for the translation sector of motion which gives us the acceleration of the center-of-mass

$$\sum \vec{F}_{ext} = M\vec{a}_{cm}. \tag{7.22}$$

For the rotational sector, we again state without proof that

$$\sum \tau_{ext-cm-z} = I_{cm}\alpha, \tag{7.23}$$

where on the RHS we have the sum of the z-components of all the external torques wrt. the center of mass, and I_{cm} is the moment of inertia about the center of mass. The interested reader may refer to Ref. [3] (not to be confused with Ref. [4]) for a proof.

7.5 Exercises

Unless stated otherwise, assume no slippage. Also, you will find it useful to use the following relation for a 1D motion

$$a = \pm\frac{dv}{dt}, \tag{7.24}$$

where we use the plus sign if the body is speeding up, and vice versa.

1. The string wrapped around a cylinder (mounted on an axle) of mass M and radius r is pulled with a horizontal force F, starting from rest. Find the angular speed after time t.

2. Consider the same system as Problem 1. Starting from rest, the string is pulled with a force of F for a distance of d. The cylinder turns without slipping. Find (i) the final angular speed of the cylinder (ii) the final speed of the cable, and (iii) the string's acceleration.

3. A string is wrapped around a cylinder of mass M and radius r, with the cylinder mounted over a pulley. The free end of the string is now attached to a block of mass m which hangs at a height h above the ground. The system is then let go from rest. Due to the hanging mass, the string unwinds without slipping. Find (i) the speed of the block and the angular speed of the cylinder when the block is about to hit the ground (ii) acceleration of the falling block (iii) tension in the string.

4. A sphere of mass m rolls without slipping so that its center of mass moves with speed v. What is its kinetic energy?

5. A circular disc of mass m rolls down an incline (with angle θ). Determine (i) the linear acceleration and (ii) friction force on the disc.

6. A circular disc rotates about a vertical axis with angular speed ω. Another identical wheel is gently glued (from a state of rest) to the first wheel. What is the common angular speed with which the two-wheel combination is finally rotating?

7. A solid ball rolls down an incline with inclination θ. What is the minimum value of the coefficient of static friction for there to be no slipping?

8. Two blocks of masses m_1 and m_2 hang on the either side of a massive pulley (which is a cylinder with radius r) with the moment of inertia I about the pivot (hence massive). The system is let go from rest. Find the acceleration of the two blocks. Hint: the tensions in the string on the either side of a massive pulley are not equal.

9. In the above problem, find the speeds of the blocks and the angular speed of the pulley when one of the blocks has descended by a height h.

10. What is the condition for the equilibrium of a light seesaw for two masses m_1 and m_2, which are at a distance d_1 and d_2 from the pivot point.

11. A rod of mass M and length l is on a table pivoted about its end. A ball with mass m strikes the rod perpendicularly with speed v at its center of mass and sticks to it. (i) What is the final angular speed of the system after the strike? (ii) What is the percentage of kinetic energy lost?

12. A rod of mass m and length l stands vertically on its pivot (which is at one of the ends of the rod). It then starts to swing downwards. Find the ω of the rod when it has completed half of a revolution (its lowest configuration)?

13. A sphere starts to roll from the top of an incline (of inclination θ) for a length l. What is the final speed of the sphere.

14. Consider Fig. 4.2 with some modifications: $\theta = 0$, and the pulley is now not massless which makes the tensions on its either side not the same (magnitude-wise). The moment of inertia of the pulley around its axle is I, and its radius is R. Find the linear acceleration of the system. There is static friction which makes the pulley rotate.

15. A spool (cylinder) of mass m has some string wound around itself. You hold the free end of the string and let the spool roll down. Find (i) the downward acceleration of the spool and (ii) the tension in the string.

Chapter 8

Gravitation

8.1 Newton's law of gravitation

A bodies of masses m_1 and m_2 attract each other with a force of Gm_1m_2/r^2, where G is the universal gravitational constant.

Also, it can be proved using calculus that insofar as the gravitational force of a spherically symmetric body is concerned, the mass of the body can be assumed to be located at its center.

8.2 Accleration due to gravity

Newton's 2nd law in conjunction with Newton's law of gravitation gives us the acceleration due to gravity of an object falling on the Earth (of mass m_E and radius r_E) near its surface

$$a = g = \frac{Gmm_E}{r_E^2}\frac{1}{m} = \frac{Gm_E}{r_E^2} = g, \tag{8.1}$$

which is independent of the nature of the falling body.

The weight of a body is the gravitational force that acts on that body, and has a magnitude mg.

8.3 Gravitational potential energy

We can derive the gravitational potential energy (at arbitrary distances) the same way we did so for the spring force and gravity (near the surface of the earth) in Sec. 5.4. Using PE $= H = -G$ (Eq. 5.16), we have (when we go from $r = r_1$ to $r = r_2$)

$$\Delta\text{PE} = \Delta H = -\Delta G = -\int_{r_1}^{r_2} -\frac{Gm_Em}{r^2}\hat{i} \cdot dr\hat{i} = \int_{r_1}^{r_2} \frac{Gm_Em}{r^2}dr \tag{8.2}$$

$$= Gm_Em\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \left(-\frac{Gm_Em}{r_2} - \left(-\frac{Gm_Em}{r_1}\right)\right), \tag{8.3}$$

where the x-axis points from the center of the earth towards the object of mass m. This motivates us to define the gravitational PE as

$$U = -\frac{Gm_Em}{r}. \tag{8.4}$$

8.4　Motion of satellites in circular orbits

Assuming uniform circular motion, we can write (with r being the distance from the satellite to the center of the earth)

$$\frac{Gmm_E}{r^2} = \frac{mv^2}{r},$$
(8.5)

$$v = \sqrt{\frac{Gm_E}{r}},$$
(8.6)

which leads to the following time period

$$T = \frac{2\pi r}{v} = \frac{2\pi r^{3/2}}{\sqrt{Gm_E}}.$$
(8.7)

8.5　Kepler's laws

1. Planets move in elliptical orbits with the Sun at the focus of the ellipse.

2. The line from the planet to the sun sweeps out equal areas in equal times.

3. Time period of planets are proportional to $3/2$ power of the semi-major axis of their orbits

8.6　Exercises

1. A rock of mass m is released from a height h above the Earth. Find the speed with which it strikes the ground. Don't assume gravity to be constant wrt. the height. The mass and radius of the Earth are M and R.

2. Two bodies move around a central star in circular orbits with constant speeds at distances r_1 and r_2. Find the ratio of their speeds.

Chapter 9

Simple harmonic motion

9.1 Mathematics of simple harmonic motion

A simple harmonic motion (SHM) is governed by the following differential equation

$$\ddot{x} = -Cx, \tag{9.1}$$

where $C > 0$ and the double dot denotes double derivative. Using $d\sin x/dt = \cos t$ and $d\cos t/dt = -\sin t$, and the chain rule for derivatives ($dy/dx = dy/dx \times dz/dz = dy/dz \times dz/dx$) one can easily check that a solution of the above equation is

$$x = A\cos(\sqrt{C}t + \phi), \tag{9.2}$$

where the amplitude A is the maximum value of x, and $\phi = -\sqrt{C}t_0$, where t_0 is the value of t when $x = A$. We state another fact that the period of Eq. 9.2 is

$$T = \frac{2\pi}{\sqrt{C}}, \tag{9.3}$$

which renders its frequency and angular frequency to be

$$f = \frac{1}{T} = \frac{\sqrt{C}}{2\pi}, \tag{9.4}$$

$$\omega = 2\pi f = \sqrt{C}. \tag{9.5}$$

9.2 Physical examples of SHM

9.2.1 Spring-mass system

Align the x-axis so that it's along the length of the spring, with the origin at the position of the mass when the spring is in its neutral state so that $x > 0$ corresponds to elongation, and $x < 0$ represents compression. With this we can write

$$F_x = ma_x = m\ddot{x} = -kx, \tag{9.6}$$

$$\ddot{x} = -\frac{k}{m}x, \tag{9.7}$$

which satisfies the SHM equation with $C = k/m$. All of the SHM results Eqs. 9.2-9.5 apply to this system with the replacement $C = k/m$.

Application of Eq. 5.20 gives us

$$\Delta\text{ME} = 0, \tag{9.8}$$

since there are no non-conservative forces in the picture, thereby showing us that the mechanical energy is conserved. So, it further follows that

$$\text{KE}_f + \text{PE}_f = \text{KE}_i + \text{PE}_i, \tag{9.9}$$

$$\frac{1}{2}mv_f^2 + \frac{1}{2}kx_f^2 = \frac{1}{2}mv_i^2 + \frac{1}{2}kx_i^2. \tag{9.10}$$

9.2.2 Pendulum

Physical pendulum

A physical pendulum is composed of a rigid body of mass m pivoted about a point. We will take the body to be planar and the oscillations to be small. With d to be the distance between the pivot and the center of mass, Eq. 7.15 lets us write

$$\tau_z = -mgd \sin \theta = -mgd\theta = I\alpha = I\ddot{\theta}, \tag{9.11}$$

$$\ddot{\theta} = -\frac{mgd}{I}\theta, \tag{9.12}$$

which upon comparison with Eqs. 9.2-9.5 (with the correspondence $\theta \to x$, $mgd/I \to C$) gives us

$$\omega = \sqrt{\frac{mgd}{I}}, \tag{9.13}$$

$$T = 2\pi\sqrt{\frac{I}{mgd}}. \tag{9.14}$$

Simple pendulum

A simple pendulum is a special case of a physical pendulum where a bob of mass m is suspended with a massless string of length l. With this, $d = l$ and $I = ml^2$. Substituting these values in Eq. 9.14 gives us the following time period

$$T = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{l}{g}}. \tag{9.15}$$

9.3 Exercises

1. A spring-mass system's period is 1 s, and amplitude is 1 m. At $t = 0$ s, the spring is in its neutral state and the block is moving in the $+x$ direction. Determine the object's location when $t = 0.5$ s.

2. A rod of mass m and length l is pivoted at its end. Find the time-period of small oscillations of this system.

3. Find the time-period of a pendulum of length l when it is taken at a height (from the surface of the Earth) which is same as the radius of the Earth r_E.

4. A block of mass m hangs from the ceiling with a spring of spring constant k. Now its displaced from its equilibrium position by pulling the block vertically downwards. Find the time-period of the resulting SHM.

5. We have a spring-mass system on a smooth table composed of a block of mass m and spring constant k in a neutral state. Another block of mass m travels towards the first block with speed v along the direction of the length of the spring and collides perfectly inelastically with the first block. Find the time period of the resulting SHM.

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